Minima of the orientation phenomena for direct *s*-*p* electron-ion excitations in dense plasmas

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Close-encounter effects on electron-ion $s \rightarrow p$ oriented excitations in dense plasmas are investigated using a first-order semiclassical straight-line trajectory method. Plasma-screened interaction potential is given by the classical nonspherical Debye-Hückel model. Without using the dipole approximation, the transition probabilities and orientation parameters are obtained, for various Debye lengths and projectile energies. Including the plasma-screening effects, the transition probabilities are increased for $1s \rightarrow 2p_{+1}$ transitions and decreased for $1s \rightarrow 2p_{-1}$ transitions. The close-encounter effects appreciably change the transition probabilities and orientation parameters for small impact parameters. It is found that the $1s \rightarrow 2p_{+1}$ transition probabilities and orientation parameters have minima that correspond to the complete $1s \rightarrow 2p_{-1}$ transitions.

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I. INTRODUCTION

Electron collisions with atoms and ions in dense and high-temperature plasmas have been extensively studied [1–7] in recent years because of their applications in many areas of physics. The orientation phenomena in ion-atom or electron-atom collisions have been actively investigated, since these phenomena provide detailed information on the mechanism of collisional excitation of target atoms and ions [7–9]. A recent experimental investigation [8] shows the possibility of the detection of radiative transitions from the excited $p_{\pm}$ states to the ground state. The orientation phenomena in dense plasma could provide detailed information about the plasma parameters. In the literature [7,9] the dipole approximation has been applied to calculate the transition probabilities and orientation parameters for the direct $s \rightarrow p$ dipole excitations. The dipole approximation is known to be valid for distant encounters, i.e., large impact parameters. However, for small impact parameters, the dipole approximation is not appropriate to evaluate the transition probabilities. Thus in this paper we investigate close-encounter effects on transition probabilities and orientation parameters for direct $s \rightarrow p$ excitation in electron–hydrogenic-ion collisions in dense plasmas. In laboratory inertial confinement fusion plasmas and astrophysical plasmas of compact objects, the plasma-coupling parameter (Γ) is much smaller than unity, so that the plasma states can be classified as weak coupling plasmas [4]. Recent investigations [3,5,6] show that the classical nonspherical Debye-Hückel model well describes the interaction potentials in weak coupling plasmas. Thus in this paper we use the nonspherical Debye-Hückel type screened-Coulomb potential for the interaction between the projectile electron and target ion. A recent investigation [7] using the straight-line (SL) trajectory with the dipole approximation shows that the plasma-screening effect appreciably reduces a favoring of the $s \rightarrow p_{-1}(m=-1)$ transition for high energy projectiles. In this paper we use the SL semiclassical trajectory, including close-encounter effects, to figure out the plasma-screening effect on the orientation parameters for small impact parameters, and to describe the motion of the projectile electron. It is found that the $1s \rightarrow 2p_{+1}$ transition probabilities and orientation parameters have minima which correspond to the complete $1s \rightarrow 2p_{-1}$ transitions. These results are quite different from the recent investigation using the dipole approximation, especially for small impact parameters. Since the relative number of coincidences for right-hand circularly polarized (RHC) and left-hand circularly polarized (LHC) photons is related to the $1s \rightarrow 2p_{\pm 1}$ excitation rates, the orientation parameters $L_{\perp}(b,\beta,a_{\Lambda})$ provide reliable temperature and density diagnostics in dense high-temperature plasmas.

In Sec. II, we derive a closed form of the semiclassical transition amplitude including the close-encounter effects for the $s \rightarrow p_{\pm 1}$ excitations in electron-ion collisions in dense plasmas using the nonspherical Debye-Hückel potential and the semiclassical SL trajectory method. In Sec. III, the transition probabilities are obtained for $1s \rightarrow 2p_{\pm 1}$ transitions including the plasma-screening effects. Also, we find that the $1s \rightarrow 2p_{+1}$ transition probabilities have minima which cannot be found in the dipole approximation. In Sec. IV we investigate the close-encounter effects on the orientation parameter for the $1s \rightarrow 2p$ excitation as a function of the impact parameter, projectile energy, and Debye length. We also find the minima of the orientation parameters. Finally, in Sec. V, a summary and discussions are given.

II. $1s \rightarrow 2p_{\pm 1}$ TRANSITION AMPLITUDES

From a first-order semiclassical approximation, the cross section for excitation from an unperturbed atomic state $n[\Psi_{nlm}(\mathbf{r})]$ to an excited state $n'[\Psi_{n'l'm'}(\mathbf{r})]$ is given by [10]

$$\sigma_{n',n} = 2\pi \int |T_{n',n}(b)|^2 b \, db \ , \tag{1}$$

where $T_{n',n}$ is the transition amplitude

$$T_{n',n} = -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, e^{i\omega_{n',n}t/\hbar}\langle n'|V(\mathbf{r},\mathbf{R})|n\rangle \ , \quad (2)$$

where $\omega_{n',n} \equiv (E_{n'} - E_n)/\hbar$, and E_n and $E_{n'}$ are the energies of atomic states n and n', respectively. For simplicity, we assume that the target is a hydrogenic ion with nuclear charge Z. Then $V(\mathbf{r},\mathbf{R})$ becomes the interaction potential between the projectile electron and target hydrogenic ion. Here $\mathbf{R}$ and $\mathbf{r}$ are the positions of the projectile electron and the bound electron, respectively. For astrophysical and laboratory plasmas, the ranges of the electron densities and the temperature are around $10^{20}-10^{23}$ cm^{-3} and 10^7-10^8 K, i.e., $\Lambda \geq 10a_Z$, where Λ is the Debye length and $a_Z(=a_0/Z)$ is the Bohr radius of the hydrogenic ion with nuclear charge Z. In these situations the plasma-coupling parameter is much smaller than unity, so that the classical Debye-Hückel model is very reliable to describe the screened Coulomb interaction. Recently, Weisheit [3] showed that a nonspherical screening Debye-Hückel potential leads to correct expressions for inelastic excitation cross sections in dense plasmas. Since then, several investigations [5,6] have been performed for the electron-impact excitation of hydrogenic ions in dense plasmas using a nonspherical Debye-Hückel potential. Due to the orthogonality of the initial and final states of the target system for inelastic scattering ($n' \neq n$), the matrix element is given by [6]

$$\langle n'|V(\mathbf{r},\mathbf{R})|n\rangle = e^2\left\langle n'\left|\frac{\exp(-|\mathbf{r}-\mathbf{R}|/\Lambda)}{|\mathbf{r}-\mathbf{R}|}\right|n\right\rangle \quad (3a)$$

$$\equiv e^2\bar{V}_{n',n} \ . \quad (3b)$$

For $1s \rightarrow 2p_{\pm 1}$ excitations, the transition matrix elements become

$$\bar{V}_{2p_{\pm 1},1s} \equiv \int d^3\mathbf{r}\, \Psi_{2p_{\pm 1}}(\mathbf{r})\frac{\exp(-|\mathbf{r}-\mathbf{R}|/\Lambda)}{|\mathbf{r}-\mathbf{R}|}\Psi_{1s}(\mathbf{r}) \ . \quad (4)$$

Using the addition theorem with the spherical harmonics Y_{lm}, the nonspherical electron-electron interaction term can be expanded [11]

$$\frac{\exp(-|\mathbf{r}-\mathbf{R}|/\Lambda)}{|\mathbf{r}-\mathbf{R}|} = \frac{4\pi}{\Lambda}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i_l(r_</\Lambda)k_l(r_>/\Lambda)Y_{lm}(\hat{r})Y_{lm}^*(\hat{R}) \ , \quad (5)$$

where i_l and k_l are the spherical modified Bessel functions and $r_>$ ($r_<$) is the smaller (larger) of r and R. Thus

$$\bar{V}_{2p_{\pm 1},1s} = \frac{\sqrt{4\pi}}{\Lambda}Y_{1\pm 1}^*(\hat{R})\{k_1(R/\Lambda)J_<(R,\Lambda) + i_1(R/\Lambda)J_>(R,\Lambda)\} \ . \quad (6)$$

Here

$$J_<(R,\Lambda) = \int_0^R r^2 dr\, R_{2p}(r)i_1(r/\Lambda)R_{1s}(r) \quad (7)$$

and

$$J_>(R,\Lambda) = \int_R^{\infty} r^2 dr\, R_{2p}(r)k_1(r/\Lambda)R_{1s}(r) \ , \quad (8)$$

where R_{1s} and R_{2p} are the radial $1s$ and $2p$ wave functions [12], respectively. Here, $J_>$ vanishes in the long range dipole approximation [7,9] ($R \gg r$). However, we shall keep this term to investigate the close-encounter effects on the transition probabilities and orientation parameters since $J_>$ would be expected to make some contribution to the transition matrix for small impact parameters. After some algebra, we obtain the $1s \rightarrow 2p_{\pm 1}$ transition matrix elements

$$\bar{V}_{2p_{\pm 1},1s} = \frac{4\sqrt{6\pi}}{(9/4-a_\Lambda)^3}\frac{Y_{1\pm 1}^*(\hat{R})}{a_Z}\left[\left\{\frac{1}{\bar{R}^2}+\frac{a_\Lambda}{\bar{R}}\right\}e^{-\bar{R}a_\Lambda} - \left\{\frac{1}{\bar{R}^2}+\frac{3/2}{\bar{R}}+\frac{9}{8}-\frac{a_\Lambda^2}{2}+\left[\frac{27}{64}-\frac{3a_\Lambda^2}{8}+\frac{a_\Lambda^4}{12}\right]\bar{R}\right\}e^{-3R/2}\right] \quad (9)$$

where $a_Z \equiv a_0/Z$ (a_0 is the Bohr radius), $a_\Lambda \equiv a_Z/\Lambda$, and $\bar{R} \equiv R/a_Z$. If we apply the dipole approximation, the terms proportional to $e^{-3\bar{R}/2}$ would be neglected. However, these terms would play an important role for close-encounter projectiles, i.e., small impact parameters. However, most of the literature [7,9] for orientation parameters, these close-encounter terms have been neglected even for small impact parameters.

To describe the projectile motion, we assume that the projectile electron is moving on a SL trajectory in the so-called natural coordinate frame [8] in which the axis of quantization z is chosen perpendicular to the collision plane. Then the position of the projectile electron can be written as

$$\mathbf{R}(t) = \mathbf{b} + \mathbf{v}t = (vt, b, 0) \ , \quad (10)$$

where v and b are the velocity of the projectile electron and the impact parameter, respectively. Here time $t=0$ is arbitrarily chosen as the instant at which the projectile electron makes its closest approach to the target ion. Under these circumstances, in $s \rightarrow p$ excitation, conservation law prohibits the $s \rightarrow p_0$ ($m=0$) transition; only $m=\pm 1$ substates ($p_{\pm 1}$) of the p level are possible. In this natural coordinate frame the spherical harmonic $Y_{1\pm 1}^*(\hat{R})$ becomes

$$Y_{1\pm 1}^*(\hat{R}) = \mp\left[\frac{3}{8\pi}\right]^{1/2}\frac{(\bar{v}t \mp i\bar{b})}{R} \ , \quad (11)$$

where $\bar{v} \equiv v/a_Z$ and $\bar{b} \equiv b/a_Z$. Then the $1s \rightarrow 2p_{\pm 1}$ transition amplitudes become

$$T_{2p_{\pm1},1s}(\beta,\bar{b},a_\Lambda)\equiv T_\pm=\mp\frac{1}{Z}\frac{2^{11}}{3^6}\frac{\beta}{(1-4a_\Lambda^2/9)^3}\times\int_0^\infty d\tau(\tau\sin\beta\tau\mp\bar{b}\cos\beta\tau)\times\left[\left\{\frac{1}{(\tau^2+\bar{b}^2)^{3/2}}+\frac{a_\Lambda}{(\tau^2+\bar{b}^2)}\right\}e^{-a_\Lambda(\tau^2+\bar{b}^2)^{1/2}}-\left\{\frac{1}{(\tau^2+\bar{b}^2)^{3/2}}+\frac{3/2}{(\tau^2+\bar{b}^2)}+\frac{(9/8-a_\Lambda^2/2)}{(\tau^2+\bar{b}^2)^{1/2}}+\left[\frac{27}{64}-\frac{3a_\Lambda^2}{8}+\frac{a_\Lambda^4}{12}\right]\right\}e^{-3(\tau^2+\bar{b}^2)^{1/2}/2}\right]. \quad (12)$$

Here we used the dimensionless time $\tau(\equiv\bar{v}t)$ and projectile energy $\varepsilon(\equiv\frac{1}{2}mv^2/Z^2Ry)$, and defined

$$\beta\equiv\omega_{2p_{\pm1},1s}a_Z/v=\frac{3}{8}\frac{1}{\sqrt{\varepsilon}}. \quad (13)$$

III. $1s\rightarrow 2p_{\pm1}$ TRANSITION PROBABILITIES

From Eq. (1), the scaled $1s\rightarrow 2p_{\pm1}$ excitation cross sections are given by

$$Z^4\sigma_\pm/\pi a_0^2=\int|\tilde{T}_\pm(\beta,\bar{b},a_\Lambda)|^2\bar{b}\,d\bar{b}, \quad (14)$$

where the scaled transition probabilities are found to be

$$\bar{b}|\tilde{T}_\pm(\beta,\bar{b},a_\Lambda)|^2=\frac{2^{23}}{3^{12}}\frac{\beta^2}{(1-4a_\Lambda^2/9)^6}\times\bar{b}\left|\int_0^\infty d\tau(\tau\sin\beta\tau\mp\bar{b}\cos\beta\tau)\times\left[\left\{\frac{1}{(\tau^2+\bar{b}^2)^{3/2}}+\frac{a_\Lambda}{(\tau^2+\bar{b}^2)}\right\}e^{-a_\Lambda(\tau^2+\bar{b}^2)^{1/2}}-\left\{\frac{1}{(\tau^2+\bar{b}^2)^{3/2}}+\frac{3/2}{(\tau^2+\bar{b}^2)}+\frac{(9/8-a_\Lambda^2/2)}{(\tau^2+\bar{b}^2)^{1/2}}+\left[\frac{27}{64}-\frac{3a_\Lambda^2}{8}+\frac{a_\Lambda^4}{12}\right]\right\}e^{-3(\tau^2+\bar{b}^2)^{1/2}/2}\right]\right|^2. \quad (15)$$

These transition probabilities include the close-encounter terms which have been neglected in the dipole approximation [7,9]. The dipole approximated transition probabilities [7] are given by

$$\bar{b}|\tilde{T}_\pm^d(\beta,\bar{b},a_\Lambda)|^2=\frac{2^{23}}{3^{12}}\frac{\beta^2}{(1-4a_\Lambda^2/9)^6}\bar{b}\left|\int_0^\infty d\tau(\tau\sin\beta\tau\mp\bar{b}\cos\beta\tau)\left\{\frac{1}{(\tau^2+\bar{b}^2)^{3/2}}+\frac{a_\Lambda}{(\tau^2+\bar{b}^2)}\right\}e^{-a_\Lambda(\tau^2+\bar{b}^2)^{1/2}}\right|^2. \quad (16)$$

Graphical comparisons of Eqs. (15) and (16) are given in Figs. 1 and 2. Here, we consider the case of $\beta=1/8$ ($\varepsilon=9$:) since the SL semiclassical approximation is likely to be valid [13] for $\varepsilon>7$, and consider two cases of the Debye lengths: $a_\Lambda=0.1$ and 0, i.e., $\Lambda/a_Z=10$ and ∞. Figures 1 and 2 show the $1s\rightarrow 2p_{+1}$ and $1s\rightarrow 1p_{-1}$ transition probabilities as functions of the impact parameter and Debye lengths. Including the plasma-screening effects, the transition probabilities are increased for $1s\rightarrow 2p_{+1}$ transitions and decreased for $1s\rightarrow 2p_{-1}$ transitions. However, the plasma-screening effect diminishes the total $1s\rightarrow 2p$ transition probability since the $1s\rightarrow 2p_{-1}$ transition dominates the $1s\rightarrow 2p_{+1}$ transition. For small impact parameters, the dipole approximated transition probabilities $\bar{b}|\tilde{T}_\pm^d(\beta,\bar{b},a_\Lambda)|^2$ blow up due to the inaccuracy of the dipole approximation for close-encounter projectiles. However, $\bar{b}|\tilde{T}_\pm(\beta,\bar{b},a_\Lambda)|^2\rightarrow 0$ as $\bar{b}\rightarrow 0$. It should be noted that $\bar{b}|\tilde{T}_+(\beta,\bar{b},a_\Lambda)|^2$ has double peaks, but on the other hand $\bar{b}|\tilde{T}_-(\beta,\bar{b},a_\Lambda)|^2$ has one peak. In other words, $\bar{b}|\tilde{T}_+|^2$ has a position $\bar{b}_0(\neq 0)$ which corresponds to $\bar{b}_0|\tilde{T}_+(\beta,\bar{b}_0,a_\Lambda)|^2=0$, i.e., zero $1s\rightarrow 2p_{+1}$ transition probability. At this impact parameter $\bar{b}_0$, the total $1s\rightarrow 2p$ excitation is completely determined by the $1s\rightarrow 2p_{-1}$ excitation. These phenomena

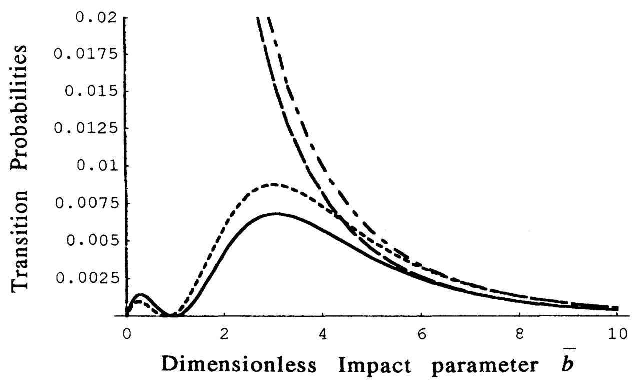

FIG. 1. The $1s \rightarrow 2p_{+1}$ transition probabilities $\bar{b}|\tilde{T}_+|^2$ for $\beta = 1/8$. The solid lines represent Eq.(15) for $a_\Lambda = 0$ (without a plasma-screening effect). The dotted lines represent Eq. (15) for $a_\Lambda = 0.1$ (with a plasma-screening effect). The dashed lines represent Eq. (16) for $a_\Lambda = 0$ (without a plasma-screening effect). The dot-dashed lines represent Eq. (16) for $a_\Lambda = 0.1$ (with a plasma-screening effect).

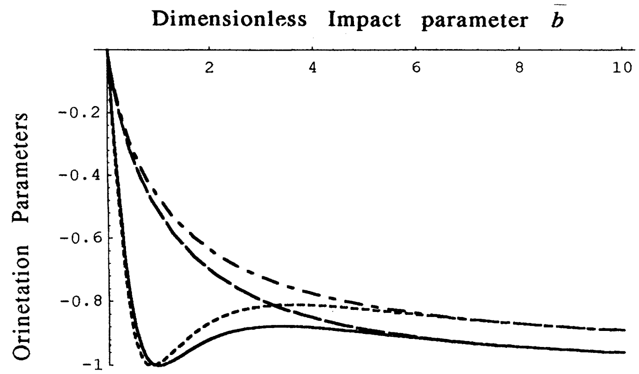

FIG. 3. The orientation parameters for $\beta = \frac{1}{8}$. The solid lines represent the orientation parameters $L_\perp(\bar{b}, \beta, a_\Lambda)$ including the close-encounter effects for $a_\Lambda = 0$ (without a plasma-screening effect). The dotted lines represent the orientation parameters including the close-encounter effects for $a_\Lambda = 0.1$ (with a plasma-screening effect). The dashed lines represent the dipole approximated orientation parameters $L_\perp^d(\bar{b}, \beta, a_\Lambda)$ for $a_\Lambda = 0$ (without a plasma-screening effect). The dot-dashed lines represent the dipole approximated orientation parameters for $a_\Lambda = 0.1$ (with a plasma-screening effect).

have never been found and explained in the literature. These are caused by the close-encounter effects $J_>$ [Eq. (9)] in $\bar{b}|\tilde{T}_+(\beta, \bar{b}, a_\Lambda)|^2$ [Eq. (15)]. Including these phenomena, we shall investigate the orientation parameter $L_\perp(\bar{b}, \beta, a_\Lambda)$ in Sec. IV.

IV. ORIENTATION PARAMETER $L_\perp(\bar{b}, \beta, a_\Lambda)$

The orientation parameter is defined [7,9] as

$$L_\perp(\bar{b}, \beta, a_\Lambda) = \frac{|T_+(\bar{b}, \beta, a_\Lambda)|^2 - |T_-(\bar{b}, \beta, a_\Lambda)|^2}{|T_+(\bar{b}, \beta, a_\Lambda)|^2 + |T_-(\bar{b}, \beta, a_\Lambda)|^2} . \quad (17)$$

This quantity $L_\perp(\bar{b}, \beta, a_\Lambda)$ is the angular momentum expectation value due to the direct $1s \rightarrow 2p_{\pm 1}$ excitations including the plasma-screening effect a_Λ. From the relationship between the orientation parameter and the degree of polarization of the emitted radiation[8], the relative number of coincidences for RHC and LHC photons is related to the orientation parameter. The orientation parameters $L_\perp(\bar{b}, \beta, a_\Lambda)$ including the close-encounter effects can be obtained by Eqs. (15) and (17). Also, the dipole approximated orientation parameters $L_\perp^d(\bar{b}, \beta, a_\Lambda)$ given by Eqs. (16) and (17). In Figs. 3 and 4 these orientation parameters are plotted as a function of the dimensionless impact parameter $\bar{b}$ for two cases of projectile energies: $\beta - \frac{1}{8}$ ($\varepsilon = 9$) and $\beta = \frac{1}{24}$ ($\varepsilon = 81$) and two cases of the Debye lengths: $a_\Lambda = 0.1$ and 0. In both figures, $L_\perp \leq 0$ for all impact parameters, i.e., the $s \rightarrow p_{-1}$ transitions, are strongly favored in all impact parameter regions. As we see in these figures, orientation parameters using the dipole approximated transition probabilities are monotonically decreasing functions of $\bar{b}$. However, the orientation parameters including the close-encounter effects have minima at $b = \bar{b}_0$. This minimum position $\bar{b}_0$

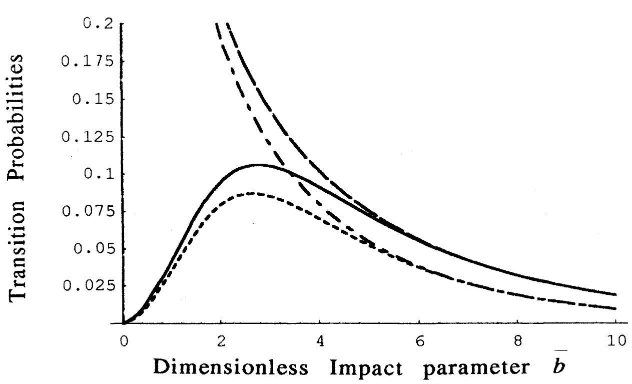

FIG. 2. The $1s \rightarrow 2p_{-1}$ transition probabilities $\bar{b}|\tilde{T}_-|^2$ for $\beta = \frac{1}{8}$. The notations are the same as Fig. 1.

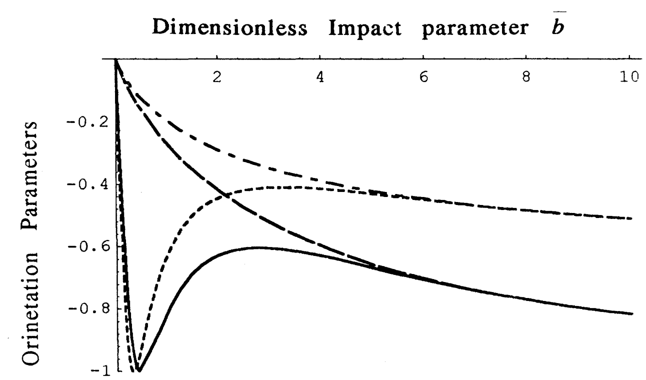

FIG. 4. The orientation parameters for $\beta = \frac{1}{24}$. The notations are the same as Fig. 3.

is shifted slightly to the nucleus with an increase of the plasma-screening effects. However, $\bar{b}_0$ is appreciably shifted to the origin as the projectile energy increases. The close-encounter effect decreases with an increase of the impact parameter. It is found that the dipole approximation is valid for $\bar{b} > 5$. This value is well above a_Z (the mean radius of the $1s$ electron). Also, it is found that the plasma-screening effect becomes less important in low energy projectiles. However, the plasma-screening effect increases with an increase of the projectile energy. The $s \to p_{-1}$ favorite is appreciably reduced as the projectile energy increases.

V. SUMMARY AND DISCUSSIONS

We investigate the close-encounter effects on electron-ion $s \to p$ oriented excitations in dense plasmas using a first-order semiclassical straight-line trajectory method. The interaction potential including the plasma screening is given by the classical nonspherical Debye-Hückel model. Including the close-encounter effects, the transition probabilities and orientation parameters are obtained for various Debye lengths and projectile energies. Including plasma-screening effects, the transition probabilities are increased for $1s \to 2p_{+1}$ transitions and decreased for $1s \to 2p_{-1}$ transitions. The close-encounter effects appreciably change the transition probabilities and orientation parameters for small impact parameters. However, the close-encounter effect decreases with an increase of the impact parameter. It is found that the $1s \to 2p_{+1}$ transition probabilities and orientation parameters have minima which correspond to the complete $1s \to 2p_{-1}$ transitions. The plasma-screening effect on the orientation parameter increases with an increase of the projectile energy. For high energy projectiles, the screening effect appreciably reduces a favorite with the $s \to p_{-1}$ transition. Since the relative number of coincidences for RHC and LHC photons is related to the $1s \to 2p_{\pm 1}$ excitation rates, the orientation parameters $L_\perp(\bar{b},\beta,a_\Lambda)$ provide reliable temperature and density diagnostics in dense high-temperature plasmas. These results provide a general description of the transition probabilities and orientation phenomena for $s \to p$ excitations in dense and high-temperature plasmas. These orientation phenomena in dense plasmas provide detailed information for the plasma parameters.

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